

# Enhanced asymptotic symmetry algebra of $AdS_3$ .

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**ABSTRACT.** A generalization of the Brown-Henneaux boundary conditions is introduced for pure gravity with negative cosmological constant in 3 dimensions. This leads to new degrees of freedom and to an enhancement of the symmetry algebra to the semi-direct product of the 2D conformal algebra with the harmonic Weyl transformations. The associated surface charge algebra now contains three central charges: the two usual Brown-Henneaux central charges and one new quantity.

# 1 Introduction

Einstein's gravity in 2+1 dimensions is an interesting toy model to understand some features of higher dimensional gravity as, in three dimensions, this theory doesn't have local degrees of freedom but still has dynamical global objects [1].

For instance, in the presence of a negative cosmological constant, there exists the famous BTZ black-hole solution [2, 3]. Those black-holes possess the same characteristics as their higher dimensional cousins like temperature or entropy. One hopes that understanding the BTZ black-holes thermodynamical properties in this simpler setup would help us in the more physically relevant cases.

Another surprise came even earlier with the study by Brown-Henneaux of the symmetry algebra of asymptotically  $AdS_3$  space-times [4]. They showed that this algebra is not the expected  $so(2, 2)$  symmetry algebra of the background  $AdS_3$  but is enhanced to the full conformal algebra in two dimensions. Furthermore the algebra acquires classical central charges with the famous value of  $c^\pm = \frac{3l}{2G}$ . Having the full 2D conformal group as a symmetry of the theory allows for the use of the powerful techniques of 2D CFT's. One of the main results is then the computation by Strominger where he was able to reproduce the Bekenstein-Hawking entropy of the BTZ black-holes using the Cardy formula and the explicit value of the Brown-Henneaux central charges [5].

In the last fifteen years, a lot has been done to further improve our understanding. For instance, using the Chern-Simons description of gravity in 3D, it was shown that gravity in this case is equivalent to a Liouville theory on the boundary [6, 7, 8, 9]. However, Liouville theory does not contain enough degrees of freedom to fully account for the entropy of the black-holes [10, 11]. More recently, a direct computation of the partition function of the theory was done but, in most cases, the results are not sensible [12, 13]. Those are some of the more recent results but, in general, we still don't have a description of the fundamental degrees of freedom of the theory.

All the results described above are strongly dependent on the Brown-Henneaux boundary conditions and the resulting asymptotic symmetry algebra. Attempts have been made to try to relax them but with no impact on the number of global degrees of freedom and no change on the asymptotic symmetry algebra [14]. A few days ago, the authors of [15] proposed a new set of chiral boundary conditions for asymptotically  $AdS_3$  space-times. Those new conditions are associated to a different problem as they only contain part of the solutions to Einstein's equations satisfying to the Brown-Henneaux boundary conditions.

In this paper, we want to present a set of boundary conditions that generalizes the one of Brown-Henneaux. Those boundary conditions describe a theory with more degrees of freedom. Moreover, there is a second enhancement of the asymptotic symmetry algebra. The new symmetry algebra is the semi-direct product of the 2D conformal algebra with

the abelian subalgebra of the harmonic Weyl transformations. At the level of the charges, the algebra is centrally extended and, in particular, we recover the usual Brown-Henneaux central charges plus one new quantity.

## 2 New asymptotic conditions

The action for gravity in 3 dimension with cosmological constant is the Einstein-Hilbert action:

$$S[g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x (R - 2\Lambda), \quad \Lambda = -\frac{1}{l^2}, \quad \mathcal{M} = \mathbb{R}^3. \quad (2.1)$$

This action is not well defined without additional boundary terms and fall-off conditions for the fields. The usual setup is given by the Brown-Henneaux boundary conditions [4]:

$$g_{AB} = r^2 \bar{\gamma}_{AB} + O(1), \quad (2.2)$$

$$g_{rA} = O(r^{-3}), \quad (2.3)$$

$$g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad (2.4)$$

where  $\bar{\gamma}_{AB}$  is a fixed metric on the cylinder at spatial infinity ( $r \rightarrow \infty$ ) and  $x^A = (\tau, \phi)$ . Choosing  $\bar{\gamma}_{AB}$  as the flat metric corresponds to asymptotically  $AdS_3$  space-times. With the metric  $\bar{\gamma}_{AB}$  fixed, the action can be supplemented with the Gibbons-Hawking boundary term to make it well defined [16, 17]:

$$S[g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} (R - 2\Lambda) + \frac{1}{16\pi G} \oint_{\partial\mathcal{M}} d^2x r^2 \sqrt{-\bar{\gamma}} (K - {}_0K) \quad (2.5)$$

where  $K$  is the trace of the extrinsic curvature of the boundary  $\partial\mathcal{M}$ . We only considered the time-like part of  $\partial\mathcal{M}$  which is the cylinder at spatial infinity. The quantity  ${}_0K = \frac{2}{l}$  acts as a counter-term to make the boundary stress energy tensor finite [18, 19, 20].

We will argue that a more general possibility is to use the same asymptotic behavior (2.2)-(2.4) but fixing only the conformal structure of the induced metric on the boundary [21, 22]:

$$g_{AB} = r^2 \gamma_{AB} + O(1), \quad \gamma_{AB} = e^{2\varphi} \bar{\gamma}_{AB} \quad (2.6)$$

where  $\varphi$  will be a dynamical field. Varying the action (2.5) now leads to

$$\begin{aligned} \delta S[g] = & \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \delta g_{\mu\nu} (-G^{\mu\nu} - \Lambda g^{\mu\nu}) \\ & + \frac{1}{16\pi G} \oint_{\partial\mathcal{M}} d^2x r^2 \sqrt{-\gamma} 2\delta\varphi \left( K - \frac{2}{l} \right). \end{aligned} \quad (2.7)$$

In order to have a well defined action, we need to impose  $K - \frac{2}{l} = o(r^{-2})$ . This corresponds to a simple change from Dirichlet to Mixed boundary conditions. On space-like

boundaries, it is equivalent to a canonical transformation but as we will see, on time-like boundaries, it changes the number of global degrees of freedom drastically.

The asymptotic conditions discussed above can be summarized by:

$$g_{rr} = \frac{l^2}{r^2} + C_{rr}r^{-4} + o(r^{-4}), \quad (2.8)$$

$$g_{rA} = O(r^{-3}), \quad (2.9)$$

$$g_{AB} = r^2\gamma_{AB} + C_{AB} + o(1), \quad (2.10)$$

$$K - {}_0K = o(r^{-2}), \quad (2.11)$$

where  $\gamma_{AB} = e^{2\varphi}\bar{\gamma}_{AB}$  and  $\bar{\gamma}$  is a fixed metric on the cylinder. The last condition is a constraint on the functions  $\varphi(x^A)$ ,  $C_{rr}(x^A)$ ,  $C_{AB}(x^C)$ :

$$\rho \equiv \gamma^{AB}C_{AB} + \frac{1}{l^2}C_{rr} = 0. \quad (2.12)$$

To describe asymptotically  $AdS_3$  space-times, the metric  $\bar{\gamma}$  has to be fixed to the flat metric:

$$\bar{\gamma}_{AB}dx^A dx^B = -d\tau^2 + d\phi^2 = -dx^+ dx^- \quad (2.13)$$

where  $x^\pm = \tau \pm \phi$  are light-cone coordinates on the cylinder. The BTZ black-hole [2, 3] satisfies those asymptotic conditions with  $\varphi = 0$  and

$$C_{\tau\tau} = 8Gl^2M, \quad C_{\phi\phi} = 0, \quad C_{\tau\phi} = 4GlJ \quad \text{and} \quad C_{rr} = 8Gl^4M, \quad (2.14)$$

or, in light-cone coordinates:

$$C_{+-} = 2Gl^2M, \quad C_{\pm\pm} = 2Gl^2 \left( M \pm \frac{J}{l} \right). \quad (2.15)$$

As we will see in section 4, any solution of Einstein's equation satisfying the Brown-Henneaux boundary conditions (2.2)-(2.4) also satisfy the supplementary condition  $\rho = 0$ . In that sense, those new boundary conditions (2.8)-(2.11) are a generalization of the usual ones.

### 3 Asymptotic symmetries

The infinitesimal diffeomorphisms leaving the conditions (2.8)-(2.11) invariant are generated by vector fields  $\xi^\mu$  satisfying

$$\mathcal{L}_\xi g_{rr} = \eta_{rr}r^{-4} + o(r^{-4}), \quad \mathcal{L}_\xi g_{rA} = O(r^{-3}), \quad (3.1)$$

$$\mathcal{L}_\xi g_{AB} = 2\omega \gamma_{AB}r^2 + \eta_{AB} + o(1), \quad (3.2)$$

$$-2\omega \gamma^{AB}C_{AB} + \gamma^{AB}\eta_{AB} + \frac{1}{l^2}\eta_{rr} = 0, \quad (3.3)$$

where  $\mathcal{L}_\xi$  is the Lie derivative. For convenience, we have denoted the induced variations on  $C_{AB}$ ,  $C_{rr}$  and  $\varphi$  by  $\eta_{AB}$ ,  $\eta_{rr}$  and  $\omega$  respectively. Equation (3.3) is coming from the variation of the supplementary condition (2.12). Equations (3.1)-(3.2) lead to

$$\begin{cases} \xi^r = -\frac{1}{2}\psi r + O(r^{-1}), \\ \xi^A = Y^A - \frac{l^2}{4r^2}\gamma^{AB}\partial_B\psi + O(r^{-4}), \end{cases} \quad (3.4)$$

where  $Y^A$  is a conformal Killing vector of  $\gamma_{AB}$ . The last equation (3.3) implies that  $\psi$  is a harmonic function: a solution of  $\Delta\psi = D_A D^A \psi = 0$  where the derivative  $D_A$  is the covariant derivative associated with  $\gamma_{AB}$ . Those conditions on  $Y^A$  and  $\psi$  depend only on the conformal structure of the cylinder:  $\bar{\gamma}_{AB}$ .

We expect the above vectors to form a closed algebra under the Lie bracket. As in [23], the vectors explicitly depend on the dynamical part of the metric  $g_{\mu\nu}$ . In that case, the usual Lie bracket has to be modified to take into account this dependance. The relevant bracket is given by:

$$[\xi_1, \xi_2]_M^\mu = [\xi_1, \xi_2]^\mu - \delta_{\xi_1}^g \xi_2^\mu + \delta_{\xi_2}^g \xi_1^\mu, \quad (3.5)$$

where we denoted by  $\delta_{\xi_1}^g \xi_2^\mu$  the change induced in  $\xi_2^\mu(g)$  due to the variation  $\delta_{\xi_1}^g g_{\mu\nu} = \mathcal{L}_{\xi_1} g_{\mu\nu}$ . Under this bracket, the vectors (3.4) satisfy:

$$\begin{aligned} [\xi_1, \xi_2]_M^r &= -\frac{1}{2}\widehat{\psi}r + O(r^{-1}), & \widehat{\psi} &= Y_1^A \partial_A \psi_2 - Y_2^A \partial_A \psi_1, \\ [\xi_1, \xi_2]_M^A &= \widehat{Y}^A - \frac{l^2}{4r^2}\gamma^{AB}\partial_B \widehat{\psi} + O(r^{-4}), & \widehat{Y}^A &= Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A. \end{aligned} \quad (3.6)$$

One can easily prove that  $\widehat{\psi}$  is again a solution of  $\Delta\widehat{\psi} = 0$ .

The set of transformations for which  $Y^A = 0 = \psi$  is an ideal of the full algebra. This is the subalgebra of pure gauge transformations; as we will see in section 5, their associated charges are zero. The asymptotic symmetry algebra is defined as the quotient of the full algebra given by (3.4) with the ideal of the pure gauge transformations [24, 25]. This quotient is parametrized by  $(Y^A, \psi)$  and the induced Lie bracket is

$$[(Y_1, \psi_1), (Y_2, \psi_2)] = (Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A, Y_1^A \partial_A \psi_2 - Y_2^A \partial_A \psi_1). \quad (3.7)$$

This algebra is the semi-direct product of the two dimensional conformal algebra with the harmonic Weyl transformations. It is a subalgebra of the Penrose-Brown-Henneaux algebra introduced in [26] which is the semi-direct product of the two dimensional conformal algebra with all Weyl transformations.

In the case of asymptotically  $AdS_3$  space-times  $\bar{\gamma}_{AB} dx^A dx^B = -dx^+ dx^-$ , the conformal Killing equation for  $Y^A$  gives as usual  $Y^+(x^+)$  and  $Y^-(x^-)$ . The harmonic equation for  $\psi$  takes the form

$$\Delta\psi = -4e^{-2\phi}\partial_+\partial_-\psi. \quad (3.8)$$

Using Fourier expansion, we easily find the general solution:

$$\psi = \sum_n \left( \psi_n^+ e^{inx^+} + \psi_n^- e^{inx^-} \right) + V\tau, \quad (3.9)$$

where  $\psi_n^\pm$  and  $V$  are constants. Denoting  $W^\pm(x^\pm) = \sum_n \psi_n^\pm e^{inx^\pm}$ , the algebra (3.7) takes the form:

$$\widehat{Y}^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm, \quad (3.10)$$

$$\widehat{W}^\pm = Y_1^\pm (\partial_\pm W_2^\pm + \frac{1}{2} V_2) - Y_2^\pm (\partial_\pm W_1^\pm + \frac{1}{2} V_1), \quad (3.11)$$

$$\widehat{V} = 0. \quad (3.12)$$

In terms of the basis vectors  $l_n^\pm, p_n^\pm$  and  $q$  defined as

$$Y^\pm(x^\pm) \partial_\pm = \sum_{n \in \mathbf{Z}} c_\pm^n l_n^\pm, \quad l_n^\pm = e^{inx^\pm} \partial_\pm, \quad (3.13)$$

$$W^\pm = \sum_{n=0} \psi_n^\pm p_n^\pm, \quad p_n^\pm = e^{inx^\pm}, \quad (3.14)$$

$$\psi = W^+ + W^- + Vq, \quad q = \tau, \quad (3.15)$$

the algebra reads

$i[l_m^\pm, l_n^\pm] = (m-n)l_{m+n}^\pm,$	$i[l_m^+, l_n^-] = 0,$	(3.16)
$i[l_m^\pm, p_n^\pm] = -np_{m+n}^\pm,$	$i[l_m^\pm, p_n^\mp] = 0,$	
$i[p_m^\pm, p_n^\pm] = 0,$	$i[p_m^+, p_n^-] = 0,$	
$i[l_m^\pm, q] = \frac{i}{2} p_m^\pm,$	$i[p_m^\pm, q] = 0.$	

The two generators  $p_0^+$  and  $p_0^-$  are identical. Each chiral copy  $(l_m, p_m)$  is the semi-direct product of a Virasoro algebra with a current algebra. One copy of this semi-direct product already appeared in the study of asymptotically warped  $AdS_3$  [27, 28, 29] and in the study of the new chiral boundary conditions for  $AdS_3$  [15].

## 4 Asymptotic solutions to the EOM

We will solve Einstein's equations asymptotically for metrics of the form (2.8)-(2.10) with the last constraint (2.12) only imposed at the end (see [30] for a similar analysis). To do that, it is useful to introduce explicitly the first order of  $g_{rA}$ :

$$g_{rr} = \frac{l^2}{r^2} + C_{rr} r^{-4} + o(r^{-4}), \quad (4.1)$$

$$g_{rA} = C_{rA} r^{-3} + o(r^{-3}), \quad (4.2)$$

$$g_{AB} = r^2 \gamma_{AB} + C_{AB} + o(1). \quad (4.3)$$

For those metrics, the Ricci tensor takes the following form

$$R_{rr} = -\frac{2}{l^2} \left( \frac{l^2}{r^2} + C_{rr} r^{-4} \right) + o(r^{-4}), \quad (4.4)$$

$$R_{rA} = \left( -\gamma^{CB} D_B C_{CA} + \gamma^{CB} D_A C_{BC} + \frac{1}{2l^2} \partial_A C_{rr} \right) r^{-3} - \frac{2}{l^2} C_{rA} r^{-3} + o(r^{-3}), \quad (4.5)$$

$$R_{AB} = -\frac{2}{l^2} (r^2 \gamma_{AB} + C_{AB}) + \gamma R_{AB} + \frac{1}{l^2} \gamma_{AB} \left( \gamma^{CD} C_{CD} + \frac{1}{l^2} C_{rr} \right) + o(1), \quad (4.6)$$

where  $\gamma R_{AB}$  is the Ricci tensor associated to the metric  $\gamma_{AB}$ . The EOM  $G_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0$  reduce asymptotically to two simple conditions:

$$\gamma R = -\frac{2}{l^2} \left( \frac{1}{l^2} C_{rr} + \gamma^{BC} C_{BC} \right), \quad (4.7)$$

$$D_B (\gamma^{BC} C_{CA} - \frac{1}{2} \delta_A^B \gamma^{CD} C_{CD}) = \frac{1}{2} \partial_A \left( \frac{1}{l^2} C_{rr} + \gamma^{BC} C_{BC} \right). \quad (4.8)$$

They are easily rewritten in term of  $\varphi$  and  $\bar{\gamma}_{AB}$ :

$$\bar{\Delta} \varphi = \frac{1}{2} \bar{R} + \frac{2e^{2\varphi}}{l^2} \rho, \quad (4.9)$$

$$\bar{D}_B \Xi_A^B = \frac{e^{2\varphi}}{2} \partial_A \rho, \quad \Xi_A^B \equiv \bar{\gamma}^{BC} C_{CA} - \frac{1}{2} \delta_A^B \bar{\gamma}^{CD} C_{CD} \quad (4.10)$$

where the barred quantities refer to the metric  $\bar{\gamma}_{AB}$ . The quantity  $\Xi_A^B$  is a symmetric traceless tensor. For our asymptotic conditions, we have to add the constrain  $\rho = 0$ . This gives us equations of motion for  $\varphi$  and  $\Xi_B^A$ :

$$\bar{\Delta} \varphi = \frac{1}{2} \bar{R}, \quad \bar{D}_B \Xi_A^B = 0. \quad (4.11)$$

Using a pure gauge transformation, one can always send a metric satisfying (4.1)-(4.3) to the Fefferman-Graham gauge-fixed form where  $g_{rr} = \frac{l^2}{r^2}$  and  $g_{rA} = 0$  [31, 32]. In that case, one can show that Einstein's equations impose

$$g_{AB} = r^2 \gamma_{AB} + \tilde{C}_{AB} + r^{-2} S_{AB}, \quad (4.12)$$

where  $\tilde{C}_{AB}$  and  $S_{AB}$  are given in term of  $\varphi$  and  $\Xi_B^A$  [33, 34, 35, 36, 23]. In that sense,  $\varphi$  and  $\Xi_B^A$  satisfying (4.11) contain all the gauge invariant degrees of freedom of the theory.

In the usual Brown-Henneaux boundary conditions, one doesn't impose  $\rho = 0$  but instead  $\gamma_{AB} dx^A dx^B = -d\tau^2 + d\phi^2$  which implies  $\varphi = 0$  and  $\bar{R} = 0$ . Inserting this in the full EOM (4.9)-(4.10), we obtain:

$$\rho = 0, \quad \bar{D}_B \Xi_A^B = 0. \quad (4.13)$$

**Theorem 4.1.** *Any solution of Einstein's equations with negative cosmological constant satisfying to the usual Brown-Henneaux boundary conditions with  $\bar{\gamma}_{AB}dx^A dx^B = -d\tau^2 + d\phi^2$  will also satisfy to the generalized boundary conditions (2.8)-(2.11).*

In that sense, we can say that our new boundary conditions are a generalization of the usual one: we are not losing any solutions. Using light-cone coordinates, we can put the EOM (4.11) in the simple form

$$\partial_+ \partial_- \varphi = 0, \quad \partial_+ \Xi_-^+ = 0, \quad \partial_- \Xi_+^- = 0. \quad (4.14)$$

Those quantities for the BTZ black-hole are given by:

$$\varphi = 0, \quad \Xi_+^- = 2Gl^2 \left( M + \frac{J}{l} \right), \quad \Xi_-^+ = 2Gl^2 \left( M - \frac{J}{l} \right). \quad (4.15)$$

We would like to emphasize the difference between the two approaches. In the Brown-Henneaux boundary conditions, one imposes  $\varphi = 0$ . The EOM then imply  $\rho = 0$  and  $D_B \Xi_A^B = 0$ . In the new boundary conditions, one imposes  $\rho = 0$  which leads the EOM (4.11) which are EOM for both  $\varphi$  and  $\Xi_A^B$ . We have new degrees of freedom in  $\varphi$ .

## 5 Surface charges

For the surface charges, we follow [37], up to a global change of sign. The technique allows us to compute the variation of the surface charges  $\delta \mathcal{Q}_\xi[h, g]$  associated to a vector field  $\xi$  under a variation of the metric  $h_{\mu\nu} = \delta g_{\mu\nu}$ . When this variation is integrable on field space [38], we can define the charges as:

$$\mathcal{Q}_\xi[g] = \int_{\gamma_s} \delta \mathcal{Q}_\xi[\delta^s g, g(\gamma_s)] \quad (5.1)$$

where the integration is done along a path  $\gamma_s$  in field space joining the background metric  $\bar{g}$  to the metric  $g$  that we are considering. For the variation of the charge, we use the expression

$$\begin{aligned} \delta \mathcal{Q}_\xi[\delta g, g] = \int_{\partial\Sigma} \frac{\sqrt{-g}}{16\pi G} (d^{n-2}x)_{\mu\nu} \left[ \xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} \right. \\ \left. + \frac{1}{2} h \nabla^\nu \xi^\mu + \frac{1}{2} h^{\nu\sigma} (\nabla^\mu \xi_\sigma - \nabla_\sigma \xi^\mu) - (\mu \leftrightarrow \nu) \right], \end{aligned} \quad (5.2)$$

where the indices are raised and lowered with the full metric  $g_{\mu\nu}$  and  $\nabla_\mu$  is the covariant derivative associated to it. We also define

$$(d^{n-k}x)_{\mu\nu} = \frac{1}{k!(n-k)!} \epsilon_{\mu\nu\alpha_1 \dots \alpha_{n-2}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{n-2}}, \quad \epsilon_{01 \dots n-1} = 1,$$



with  $n = 3$  and the surface of integration  $\partial\Sigma$  is taken to be a circle on the cylinder at spatial infinity  $C_\infty$ .

A straightforward computation leads to

$$\begin{aligned} \delta\mathcal{Q}_\xi[h, g] = & \frac{1}{16\pi Gl} \delta \int_{C_\infty} \epsilon_{AB} dx^B \sqrt{-\bar{\gamma}} \left[ l^2 \bar{\gamma}^{AC} (\partial_C \psi \varphi - \psi \partial_C \varphi) + 2Y^A \bar{\gamma}^{CD} C_{CD} - 2Y^C \bar{\gamma}^{AD} C_{CD} \right] \\ & + \frac{1}{16\pi Gl} \int_{C_\infty} \epsilon_{AB} dx^B \sqrt{-\bar{\gamma}} Y^A \left( \frac{1}{l^2} e^{2\varphi} \delta C_{rr} - 2\delta\varphi \bar{\gamma}^{CD} C_{CD} \right) \end{aligned} \quad (5.3)$$

where we used  $\epsilon_{AB} = -\epsilon_{rAB}$ . The second line contains a non-integrable term that is easily removed using the variation of the constraint (2.12). The final result, after integrating in field space, is given by:

$$\mathcal{Q}_\xi[g] = \frac{1}{16\pi Gl} \int_{C_\infty} \epsilon_{AD} dx^D \sqrt{-\bar{\gamma}} \left[ l^2 (\bar{D}^A \psi \varphi - \psi \bar{D}^A \varphi) - 2Y^B \Xi_B^A \right], \quad (5.4)$$

where we raised and lowered indices with  $\bar{\gamma}_{AB}$  and its inverse. We see that the 2 relevant dynamical quantities for the charges are, as expected,  $\varphi$  and  $\Xi_B^A$ . We normalized our charges using the BTZ black-hole with  $M = 0 = J$  as a background (or, using (4.15),  $\varphi = 0$  and  $\Xi_B^A = 0$ ). The charges depend only on the leading orders of  $\xi$ : the pure gauge transformations  $\xi^r = O(r^{-1})$  and  $\xi^A = O(r^{-4})$  give zero.

In the asymptotically  $AdS_3$  case, with  $\bar{\gamma}_{AB} dx^A dx^B = -d\tau^2 + d\phi^2$  and  $C_\infty$  being the circle at  $\tau$  constant, we obtain:

$$\mathcal{Q}_\xi[g] = \frac{1}{16\pi Gl} \int_0^{2\pi} d\phi \left[ l^2 (\psi \partial_\tau \varphi - \partial_\tau \psi \varphi) + 2Y^\tau \Xi_{\tau\tau} + 2Y^\phi \Xi_{\tau\phi} \right]. \quad (5.5)$$

The last two terms are the usual contribution coming from the two Virasoro algebras. Using the time translation and angular rotation symmetry vectors  $Y = \frac{1}{l} \partial_\tau$  and  $Y = \partial_\phi$ , we can evaluate the mass and the angular momentum of the BTZ black-hole:

$$\mathcal{Q}_{\frac{1}{l}\partial_\tau}[g_{BTZ}] = M, \quad \text{and} \quad \mathcal{Q}_{\partial_\phi}[g_{BTZ}] = J. \quad (5.6)$$

In light-cone coordinates and using the parametrization of the asymptotic symmetry group in term of  $(Y^\pm, W^\pm, V)$  introduced in section 3, the charges (5.5) can be rewritten as

$$\begin{aligned} \mathcal{Q}_\xi[g] \approx & \frac{1}{8\pi Gl} \int_0^{2\pi} d\phi \left[ l^2 (W^+ \partial_+ \varphi^+ + W^- \partial_- \varphi^-) + Y^+ \Xi_{++} + Y^- \Xi_{--} \right] \\ & + \frac{\alpha}{16\pi Gl} \int_0^{2\pi} d\phi (W^+ + W^-) - \frac{V}{16\pi Gl} \int_0^{2\pi} d\phi (\varphi^+ + \varphi^-). \end{aligned} \quad (5.7)$$

To obtain this result, we used some integration by parts and solved the EOM (4.14) with  $\varphi = \varphi^+(x^+) + \varphi^-(x^-) + \alpha\tau$ ,  $\alpha$  being a constant.

## 6 Centrally extended algebra

As in [37, 38], we expect the charges built in the previous section to form a representation of the asymptotic symmetry algebra, or more precisely, that

$$\left[ \mathcal{Q}_{\xi_1}[g], \mathcal{Q}_{\xi_2}[g] \right] \equiv \delta \mathcal{Q}_{\xi_1}[\mathcal{L}_{\xi_2}g, g] \approx \mathcal{Q}_{[\xi_1, \xi_2]_M}[g] + K_{\xi_1, \xi_2}, \quad (6.1)$$

where  $K_{\xi_1, \xi_2}$  is a possible central extension.

For vectors satisfying (3.4),  $\mathcal{L}_{\xi}g_{AB}$  leads to the following variations:

$$\delta_{\xi}\varphi = Y^A \partial_A \varphi + \frac{1}{2} \bar{D}_A Y^A - \frac{1}{2} \psi, \quad (6.2)$$

$$\begin{aligned} \delta_{\xi} \Xi_{AB} &= \mathcal{L}_Y \Xi_{AB} - \frac{l^2}{2} \bar{D}_A \partial_B \psi \\ &\quad + \frac{l^2}{2} (\partial_A \varphi \partial_B \psi + \partial_B \varphi \partial_A \psi - \bar{\gamma}_{AB} \bar{D}^C \varphi \partial_C \psi). \end{aligned} \quad (6.3)$$

Using those in (5.4) and integration by parts, we get

$$\begin{aligned} \delta \mathcal{Q}_{\xi_1}[\mathcal{L}_{\xi_2}g, g] &= \frac{1}{16\pi G l} \int_{C_{\infty}} \epsilon_{AD} dx^D \sqrt{-\bar{\gamma}} \left[ l^2 (\bar{D}^A \hat{\psi} \varphi - \hat{\psi} \bar{D}^A \varphi) - 2 \hat{Y}^B \Xi_B^A \right. \\ &\quad + \frac{l^2}{2} (\psi_1 \bar{D}^A \psi_2 - \psi_2 \bar{D}^A \psi_1) + l^2 (Y_1^B \bar{D}^A \partial_B \psi_2 - Y_2^B \bar{D}^A \partial_B \psi_1) \\ &\quad \left. - l^2 \psi_1 Y_2^A \bar{D}_B \bar{D}^B \varphi + \frac{l^2}{2} Y_2^A \psi_1 \bar{R} - 2 Y_1^B Y_2^A \bar{D}_E \Xi_B^E \right]. \end{aligned} \quad (6.4)$$

Onshell, this reproduces (6.1) with

$$K_{\xi_1, \xi_2} = \frac{l}{16\pi G} \int_{C_{\infty}} \epsilon_{AD} dx^D \sqrt{-\bar{\gamma}} \left[ \frac{1}{2} \psi_1 \bar{D}^A \psi_2 + Y_1^B \bar{D}^A \partial_B \psi_2 - (1 \leftrightarrow 2) \right], \quad (6.5)$$

which satisfies the cyclic identity:

$$K_{[\xi_1, \xi_2]_M, \xi_3} + K_{[\xi_2, \xi_3]_M, \xi_1} + K_{[\xi_3, \xi_1]_M, \xi_2} = 0. \quad (6.6)$$

As expected, the algebra closes and we obtain a non-zero central extension. However, as one can see clearly in the expression (6.5), there are no central terms in the conformal subalgebra parametrized by  $Y^A$ .

In the asymptotically  $AdS_3$  case, using equation (5.7) for the charges and some integration by parts, we obtain

$$\begin{aligned} \left[ \mathcal{Q}_{\xi_1}[g], \mathcal{Q}_{\xi_2}[g] \right] &\approx \mathcal{Q}_{[\xi_1, \xi_2]_M}[g] + K_{\xi_1, \xi_2}^+ + K_{\xi_1, \xi_2}^- + K_{\xi_1, \xi_2}^0, \\ K_{\xi_1, \xi_2}^+ &= \frac{l}{16\pi G} \int_0^{2\pi} d\phi (W_1^+ \partial_+^2 Y_2^+ - W_2^+ \partial_+^2 Y_1^+ - W_1^+ \partial_+ W_2^+), \\ K_{\xi_1, \xi_2}^- &= \frac{l}{16\pi G} \int_0^{2\pi} d\phi (W_1^- \partial_-^2 Y_2^- - W_2^- \partial_-^2 Y_1^- - W_1^- \partial_- W_2^-), \\ K_{\xi_1, \xi_2}^0 &= \frac{l V_1}{32\pi G} \int_0^{2\pi} d\phi (W_2^+ + W_2^-) - \frac{l V_2}{32\pi G} \int_0^{2\pi} d\phi (W_1^+ + W_1^-), \end{aligned} \quad (6.7)$$

where  $K_{\xi_1, \xi_2}^{\pm, 0}$  are the central extensions. Expanding this result in term of the charges  $(L_m^\pm, P_m^\pm, Q)$  associated to the basis  $(l_m^\pm, p_m^\pm, q)$  introduced in section 3, we obtain explicitly

$$\begin{array}{ll} i[L_m^\pm, L_n^\pm] = (m-n)L_{m+n}^\pm, & i[L_m^+, L_n^-] = 0, \\ i[L_m^\pm, P_n^\pm] = -nP_{m+n}^\pm + \frac{l}{8G}im^2\delta_{m+n,0}, & i[L_m^\pm, P_n^\mp] = 0, \\ i[P_m^\pm, P_n^\pm] = -\frac{l}{8G}m\delta_{m+n,0}, & i[P_m^+, P_n^-] = 0, \\ i[L_m^\pm, Q] = \frac{i}{2}P_m^\pm, & i[P_m^\pm, Q] = -i\frac{l}{16G}\delta_{m,0}. \end{array} \quad (6.8)$$

As we saw earlier, adding dynamics to the conformal factor of the boundary metric sends the Virasoro central charges to zero. This effect is similar to the one described in [39] where Liouville theory is coupled to gravity in 2 dimension.

However, as we will see in the next section, there are more than one 2D conformal algebra hidden in this algebra and it is possible to recover the usual Brown-Henneaux central extension.

## 7 Brown-Henneaux central charges recovered

At first sight, the final algebra (6.8) is not very promising. The central charges in the Virasoro algebras are a key point of the various results obtained in asymptotically  $AdS_3$  space-times and we lost them. However, as the boundary conditions studied are a generalization of the usual ones, the central charges must be hidden somewhere.

The answer comes by studying the exact Killing vectors of the background  $AdS_3$ . The original Virasoro algebras studied by Brown-Henneaux are built on the Killing vectors of  $AdS_3$ , in the sense that,  $l_{-1}^\pm, l_0^\pm$  and  $l_1^\pm$  are the generators of the  $so(2, 2)$  algebra leaving  $AdS_3$  invariant. As we will show in the following, the Virasoro generators present in the basis used to write our algebra (3.16) do not satisfy this property. Nevertheless, it is possible to recover it by doing a change of basis of the algebra. This will also reproduce the usual Brown-Henneaux result for the central extension of the 2D conformal subalgebra.

The  $AdS_3$  metric is given by:

$$ds^2 = -\left(\frac{r^2}{l^2} + 1\right)dt^2 + \frac{1}{\frac{r^2}{l^2} + 1}dr^2 + r^2d\phi^2. \quad (7.1)$$

In our asymptotic expansion, it corresponds to:

$$\varphi = 0, \quad \Xi_{++} = -\frac{l^2}{4}, \quad \Xi_{--} = -\frac{l^2}{4}. \quad (7.2)$$

The Killing vectors of  $AdS_3$  are asymptotic symmetries that preserve those three quanti-

ties. Using the variations (6.2) and (6.3), we obtain the following equations:

$$\delta_\xi \varphi = \partial_+ Y^+ + \partial_- Y^- - \psi = 0, \quad (7.3)$$

$$\delta_\xi \Xi_{++} = -\frac{l^2}{2} (\partial_+ Y^+ + \partial_+^2 \psi) = 0, \quad (7.4)$$

$$\delta_\xi \Xi_{--} = -\frac{l^2}{2} (\partial_- Y^- + \partial_-^2 \psi) = 0. \quad (7.5)$$

It is obvious that the  $l_{\pm 1}^\pm$  defined in section 3 are not solutions to those equations: they are not Killing vectors of  $AdS_3$ . The general solution is given by the set of vectors  $(Y^A, \psi = \partial_A Y^A)$  with  $Y^A$  satisfying  $\partial_\pm Y^\pm = \partial_\pm^3 Y^\pm$ . In terms of our generators  $(l_m^\pm, p_m^\pm)$  the Killing vectors of  $AdS_3$  are given by:

$$l_0^\pm, \quad l_1^\pm + i p_1^\pm, \quad l_{-1}^\pm - i p_{-1}^\pm. \quad (7.6)$$

We can build two full Virasoro algebras on those vectors as follows:

$$\tilde{l}_m^\pm \equiv l_m^\pm + i m p_m^\pm. \quad (7.7)$$

The generators  $(\tilde{l}_m^\pm, p_m^\pm, q)$  form a new basis of our asymptotic symmetry algebra for which the commutators (3.16) take the same form:

$$\begin{aligned} i[\tilde{l}_m^\pm, \tilde{l}_n^\pm] &= (m-n)\tilde{l}_{m+n}^\pm, & i[\tilde{l}_m^+, \tilde{l}_n^-] &= 0, \\ i[\tilde{l}_m^\pm, p_n^\pm] &= -n p_{m+n}^\pm, & i[\tilde{l}_m^\pm, p_n^\mp] &= 0, \\ i[p_m^\pm, p_n^\pm] &= 0, & i[p_m^+, p_n^-] &= 0, \\ i[\tilde{l}_m^\pm, q] &= \frac{i}{2} p_m^\pm, & i[p_m^\pm, q] &= 0. \end{aligned} \quad (7.8)$$

On the level of the associated charges  $(\tilde{L}_m^\pm = L_m^\pm + i m P_m^\pm, P_m^\pm, Q)$ , we recover the usual result for the Virasoro central charges:

$\begin{aligned} i[\tilde{L}_m^\pm, \tilde{L}_n^\pm] &= (m-n)\tilde{L}_{m+n}^\pm + \frac{c^\pm}{12} m^3 \delta_{m+n,0}, \\ i[\tilde{L}_m^\pm, P_n^\pm] &= -n P_{m+n}^\pm, \\ i[P_m^\pm, P_n^\pm] &= k m \delta_{m+n,0}, \\ i[\tilde{L}_m^\pm, Q] &= \frac{i}{2} P_m^\pm, \end{aligned}$	$\begin{aligned} i[\tilde{L}_m^+, \tilde{L}_n^-] &= 0, \\ i[\tilde{L}_m^\pm, P_n^\mp] &= 0, \\ i[P_m^+, P_n^-] &= 0, \\ i[P_m^\pm, Q] &= \frac{i k}{2} \delta_{m,0}. \end{aligned}$
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(7.9)

The central extension is now characterized by three numbers: the Brown-Henneaux central charges  $c^\pm = \frac{3l}{2G}$  and one new quantity  $k = -\frac{l}{8G}$ . This new extension in the current part of the algebra  $(P_m^\pm, Q)$  can not be absorbed by a redefinition of the generators. Moreover, it is only characterized by one number because of  $P_0^+ = P_0^-$  and the Jacobi identity. The factor of  $m^3$  is coming from our normalization for  $\tilde{L}_0^\pm$ : using  $AdS_3$  as a background would lead to the standard  $m^3 - m$ .

Similar algebras appear in the study of higher spin gravity in three dimensions [40, 41]. As in their case and in the result of [15], the central charges  $c^\pm$  and  $k$  have opposite signs which will lead to non unitary representations [29].

## 8 Conclusions

The boundary conditions studied in this paper are a generalization of the usual Brown-Henneaux boundary conditions. Those extended boundary conditions lead to a second enhancement of the asymptotic symmetry algebra to the semi-direct product of the conformal algebra in two dimensions with the harmonic Weyl transformations. At the level of the charges, the algebra is centrally extended and we recover the Brown-Henneaux central charges  $c^\pm = \frac{3l}{2G}$  plus one new number  $k = -\frac{l}{8G}$ .

Those boundary conditions give us two interesting things. There are more degrees of freedom which would maybe account for what we are missing in our understanding of gravity in three dimensions. The second improvement is a bigger symmetry algebra. This would give us more tools to control and understand the theory.

For the future, it would be interesting to see how the results obtained in the study of asymptotically  $AdS_3$  space-times change. As most of those results rely heavily on the result of Brown-Henneaux, a change in boundary conditions can have a strong impact.

A striking difference with the new chiral boundary conditions of [15] is the size of the algebra. It might be possible to generalize those chiral boundary conditions and enhance the associated asymptotic symmetry algebra.

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